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# AN EXPLICIT FORMULA FOR THE LIMITING OPTIMAL GAIN IN THE FULL INFORMATION DURATION PROBLEM (Mathematics of Decision-making under uncertainty)

AUTHOR(S):

Mazalov, Vladimir; Tamaki, Mitsushi

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# AN EXPLICIT FORMULA FOR THE LIMITING OPTIMAL GAIN IN THE FULL INFORMATION DURATION PROBLEM

Vladimir Mazalov Karel'ian Research Center

玉置 光司 (Mitsushi Tamaki) Faculty of Business Administration, Aichi University

## 1 Introduction and summary

We consider here a full-information model for the duration problem [Ferguson, Hardwik, Tamaki 1992] with horizon  $n$  tending to infinity. Our objective here is to determine the asymptotics for the optimal gain  $V(n)$ .

Suppose that  $X_1, X_2, \dots$  are i.i.d. random variables, uniformly distributed on  $[0, 1]$ , where  $X_n$  denotes the value of the object at the  $n$ -th stage from the end. We call an object relatively best if it possesses the largest value among those observed so far. The task is to select a relatively best object with the view of maximizing the duration it stays relatively best. Let  $v(x, n)$  denote the optimal expected return when there are  $n$  objects yet to be observed and the present maximum of past observations is  $x$ . Notice that  $V(n) = v(0, n)$ .

The Optimality Equation for  $v(x, n)$  has form

$$v(x, n) = xv(x, n-1) + \int_x^1 \max\{w(t, n), v(t, n-1)\} dt, \quad v(x, 0) = 0, \quad (1)$$

where  $w(x, n)$  denote the expected payoff given that the  $n$ th object from the last is a relatively best object of value  $X_n = x$  and we select it.

$$w(x, n) = 1 + x + x^2 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}. \quad (2)$$

Denote the point of intersection of functions  $v(x, n-1)$  and  $w(x, n)$  as  $x_n$ . It exists and unique because  $w(x, n)$  are increasing in  $x$  for every  $n$  and  $v(x, n)$  are nonincreasing in  $x$  for every  $n$ , and  $w(0, n) = 1 \leq \int_0^1 w(t, n) dt \leq v(0, n)$ , and  $w(1, n) = n > 0 = v(1, n-1)$ .

So, we can rewrite (1) in the form

$$v(x, n) = xv(x, n-1) + \int_x^{x_n} v(t, n-1) dt + \int_{x_n}^1 w(t, n) dt, \quad 0 \leq x \leq x_n \quad (3)$$

and

$$v(x, n) = xv(x, n-1) + \int_x^1 w(t, n) dt, \quad x_n \leq x \leq 1. \quad (4)$$

If we stop the selection with a relatively best object  $X_n = x$ , we receive  $w(x, n)$ . If we continue and select the next relatively best object, we expect to receive

$$\begin{aligned} u(x, n) &= \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t, n-k) dt \\ &= \sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k} (1-x^j)/j. \end{aligned} \quad (5)$$

The problem is monotone [Ferguson et al, 1992], so the one-stage look-ahead rule (OLA) is optimal here and prescribes stopping if  $w(x, n) \geq u(x, n)$ ; that is, if

$$\sum_{k=1}^n x^{k-1} \left( 1 - \sum_{j=1}^{n-k} (1-x^j)/j \right) \geq 0. \quad (6)$$

It is equivalent that we stop selection on the step  $n$  if the relatively best object has value  $X_n \geq x_n$  (see the fig. 1) with  $x_n$  as the solution of the equation (6).

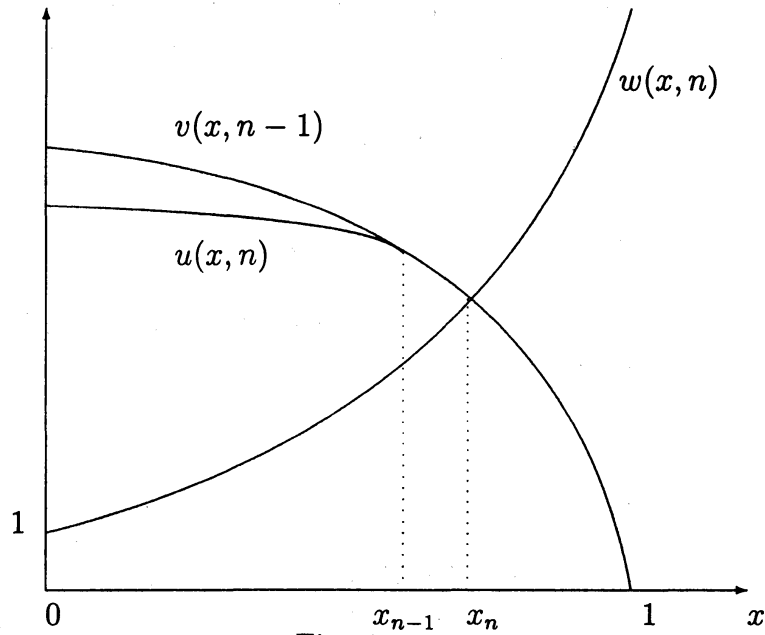


Fig. 1

According to [Porosinski, 1987],  $x_n$  written as  $x_n = 1 - z_n/n$  satisfies the equation

$$\sum_{k=1}^n \left( 1 - \frac{z_n}{n} \right)^{k-1} \left( 1 - \sum_{j=1}^{n-k} \left( 1 - \left( 1 - \frac{z_n}{n} \right)^j \right) / j \right) = 0,$$

and from here  $z_n$  must converge to a constant,  $z_n \rightarrow z$ , where  $z \approx 2.11982$  satisfies the integral equation

$$\int_0^1 e^{-zv} \left[ 1 - \int_0^{1-v} \frac{1 - e^{-zu}}{u} du \right] dv = 0.$$

**Lemma 1.**  $V(n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Proof.** It follows immediately from the inequality  $v(x, n) \geq u(x, n+1)$  for every  $x, n$  and  $u(0, n) = \sum_{j=1}^{n-1} 1/j \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Lemma 2.** There exists a constant  $C$  such that  $V(n) \leq Cn$  for all  $n$ .

**Proof.** Suppose that it is true for some  $n-1$ . Then from (3)-(4) it follows that for every  $x$   $v(x, n) \leq C(n-1) + \epsilon_n$  where  $\epsilon_n = \int_{x_n}^1 w(t, n)dt$ . Because  $z_n \rightarrow z$ , the sequence

$$\begin{aligned} \epsilon_n &= \sum_{j=1}^n \frac{1 - x_n^j}{j} \\ &= \sum_{j=1}^n \frac{1 - (1 - \frac{z_n}{n})^j}{j} \\ &= \sum_{j=1}^n \frac{1 - (1 - \frac{z_n}{n})^{\frac{j}{n}n}}{j/n} \frac{1}{n} \end{aligned}$$

converges to  $\int_0^1 (1 - e^{-zx})/x dx \approx 1.3700$  and, consequently, it is bounded by some constant  $e$ , i.e.  $\epsilon_n \leq e$ . Hence, if we choose  $C \geq e$  we obtain  $v(x, n) \leq Cn$  for all  $x$ .

**Remark.** So, the function  $V(n)$  tends to infinity not slower than  $\log n$  and not faster than  $Cn$ .

**Lemma 3.** Function  $u(x, n)$  satisfies the equation

$$u(x, n) = xu(x, n-1) + \int_x^1 w(t, n-1)dt, \quad u(x, 1) = 0. \quad (7)$$

**Proof.** It follows from direct calculations

$$\begin{aligned} u(x, n) - xu(x, n-1) &= \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t, n-k)dt - \sum_{k=1}^{n-2} x^k \int_x^1 w(t, n-1-k)dt \\ &= \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t, n-k)dx - \sum_{k=2}^{n-1} x^{k-1} \int_x^1 w(t, n-k)dx \\ &= \int_x^1 w(t, n-k)dx. \end{aligned}$$

Let us introduce two new functions

$$y(x, n) = v(x, n) - u(x, n+1), \quad \Delta_n(x) = u(x, n) - w(x, n).$$

In the interval  $[0, x_n]$  both functions are non-negative and  $\Delta_n(x_n) = 0$ . According to (3) and (7),  $y(x, n)$  satisfies the equation

$$y(x, n) = xy(x, n-1) + \int_x^{x_n} [y(t, n-1) + \Delta_n(t)] dt, \quad 0 \leq x \leq x_n, \quad (8)$$

and  $y(x, n) = 0$ , for  $x \geq x_n$ . Also, notice that  $y(x, 1) = y(x, 2) = 0$  (because  $x_1 = x_2 = 0$ ) and  $y(x, 3) = \int_x^{x_3} \Delta_3(t) dt$ , where  $\Delta_3(x) = 1/2 - x - 5/2x^2$ .

**Lemma 4.** *Function  $\Delta_n(x)$  satisfies the equations*

$$\Delta_n(x) = x\Delta_{n-1}(x) + \int_x^1 w(t, n-1) dt - 1, \quad \Delta_1(x) = -1. \quad (9)$$

$$\Delta_{n+1}(x) - \Delta_n(x) = \sum_{j=1}^n \frac{x^{n-j} - x^n}{j} - x^n. \quad (10)$$

**Proof.** (9) follows from Lemma 3 and the simple identity

$$w(x, n) = xw(x, n-1) + 1.$$

From (9) we obtain

$$\Delta_{n+1}(x) - \Delta_n(x) = x \left[ \Delta_n(x) - \Delta_{n-1}(x) \right] + \frac{1 - x^n}{n}.$$

After  $n$  iterations we have

$$\Delta_{n+1}(x) - \Delta_n(x) = \sum_{j=1}^n \frac{x^{n-j} - x^n}{j} + x^n [\Delta_1(x) - \Delta_0(x)].$$

With  $\Delta_1(x) - \Delta_0(x) = -1$  it proves (10).

Differentiating (8) in  $x$  we obtain

$$y'(x, n) = xy'(x, n-1) - \Delta_n(x), \quad 0 \leq x \leq x_n,$$

and, consequently,  $y'(x, 3) = -\Delta_3(x)$ ,

$$y'(x, 4) = \begin{cases} -x\Delta_3(x) - \Delta_4(x), & \text{if } 0 \leq x \leq x_3 \\ -\Delta_4(x), & \text{if } x_3 < x \leq x_4 \end{cases}$$

and

$$y'(x, n) = - \sum_{j=i}^n x^{n-j} \Delta_j(x), \quad x_{i-1} \leq x \leq x_i, i = 3, 4, \dots, n. \quad (11)$$

In integral form (11) is

$$y(x, n) = \sum_{j=i}^n \int_x^{x_j} t^{n-j} \Delta_j(t) dt, \quad x_{i-1} \leq x \leq x_i, i = 3, 4, \dots, n. \quad (12)$$

Letting in (12)  $x = 0$  we obtain

**Lemma 5.**

$$y_n = y(0, n) = \sum_{j=3}^n \int_0^{x_j} t^{n-j} \Delta_j(t) dt. \quad (13)$$

Consider now the difference  $r_n = y_{n+1} - y_n$ . Using (13) we can represent it as a sum of two expressions

$$\begin{aligned} r_n &= y_{n+1} - y_n \\ &= \sum_{j=3}^n \int_0^{x_j} t^{n-j} [\Delta_{j+1}(t) - \Delta_j(t)] dt + \sum_{j=3}^{n+1} \int_{x_{j-1}}^{x_j} t^{n-j+1} \Delta_j(t) dt \end{aligned} \quad (14)$$

Using (10) the first sum in (14) can be rewritten in the form

$$\sum_{j=3}^n \int_0^{x_j} t^{n-j} \left[ \sum_{i=1}^j \frac{t^{j-i} - t^j}{i} - t^j \right] dt = \sum_{j=3}^n \left\{ \sum_{i=1}^j \left[ \frac{x_j^{n-i+1}}{n-i+1} - \frac{x_j^{n+1}}{n+1} \right] \frac{1}{i} - \frac{x_j^{n+1}}{n+1} \right\}. \quad (15)$$

As  $n \rightarrow \infty$  letting  $x_n = 1 - z_n/n$  we obtain that (15) converges to the integral

$$C_0 = \int_0^1 e^{-\frac{z}{u}} \left[ \int_0^u \left( \frac{e^{\frac{zv}{u}} - 1}{v} + \frac{e^{\frac{zv}{u}}}{1-v} \right) dv - 1 \right] du. \quad (16)$$

Let us prove that the second sum in (14) tends to zero. Firstly notice that  $\Delta_{j-1}(x) \leq 0$  for  $x \geq x_{j-1}$ , consequently, from Lemma 4 it follows that

$$\Delta_j(x) \leq \int_x^1 w(t, j-1) dt - 1 \leq \sum_{i=1}^{j-1} \frac{1}{i}, \quad x \in [x_{j-1}, x_j].$$

Hence,

$$\begin{aligned} 0 &\leq \sum_{j=3}^{n+1} \int_{x_{j-1}}^{x_j} t^{n-j+1} \Delta_j(t) dt \\ &\leq \sum_{j=3}^{n+1} \sum_{i=1}^{j-1} \int_{x_{j-1}}^{x_j} \frac{t^{n-j+1}}{i} dt \\ &= \sum_{i=1}^n \frac{1}{i} \sum_{j=i+1}^{n+1} \left[ \frac{x_j^{n-j+2} - x_{j-1}^{n-j+2}}{n-j+2} \right] \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^n \frac{1}{i} \sum_{j=i+1}^{n+1} \left[ x_j^{n-j+1} (x_j - x_{j-1}) \right] \\
&= \sum_{i=1}^n \frac{1}{i} \sum_{j=i+1}^{n+1} \left[ \left(1 - \frac{z_j}{j}\right)^{n-j+1} \left(\frac{z_{j-1}}{j-1} - \frac{z_j}{j}\right) \right].
\end{aligned} \tag{17}$$

Using the inequality  $1 - x \leq \exp(-x)$ ,  $\forall x$ , (17) is not larger than

$$\begin{aligned}
&\sum_{i=1}^n \frac{1}{i} \sum_{j=i+1}^{n+1} \left[ \exp\left\{-\frac{z_j}{j}(n-j+1)\right\} \left(\frac{z_{j-1}}{j-1} - \frac{z_j}{j}\right) \right] \\
&\leq \exp\{Z\} \sum_{i=1}^n \frac{1}{i} \sum_{j=i+1}^{n+1} \left[ \exp\left\{-\frac{z_j(n+1)}{j}\right\} \left(\frac{z_{j-1}}{j-1} - \frac{z_j}{j}\right) \right],
\end{aligned} \tag{18}$$

where  $Z = \sup z_j$ . The last sum (18) converges as  $n \rightarrow \infty$  to zero, because  $z_n \rightarrow z$  and corresponding for internal sum integral  $\int_{i+1}^{n+1} \exp\{-\frac{z(n+1)}{t}\} z/t^2 dt$  is not larger than  $\exp\{-z\}/(n+1)$ .

Summarizing all, we obtain the next result.

**Theorem 1.** *For large  $n$*

$$\frac{V_n}{n} \rightarrow C_0,$$

where  $C_0$  is the value of the integral (16).

Proof follows immediately from  $\lim_{n \rightarrow \infty} V_n/n = \lim_{n \rightarrow \infty} v(0, n)/n = \lim_{n \rightarrow \infty} [y_n + u(0, n+1)]/n = \lim_{n \rightarrow \infty} [y_n + \sum_{j=1}^n 1/j]/n = C_0$ . Package *Mathematica* calculates  $C_0 \approx 0.4351708$ .

## REFERENCES

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